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Holography, singularities on orbifolds and 4D $N=2$ SQCD

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Abstract: Type-II string theory compactified on a Calabi-Yau manifold, with a singularity modeled by a hypersurface in an orbifold, is considered. In the limit of vanishing string coupling, one expects a non gravitational theory concentrated at the singularity. It is proposed that this theory is holographicly dual to a family of "non-critical" superstring vacua, generalizing a previous proposal for hypersurfaces in flat space. It is argued that a class of such singularities is relevant for the study of non-trivial IR fixed points that appear in the moduli space of four-dimensional $N=2$ SQCD: $\mathrm{SU}(N_c)$ gauge theory with matter in the fundamental representation. This includes the origin in the moduli space of the $\mathrm{SU}(N_c)$ gauge theory with $N_f = 2N_c$ fundamentals. The 4D IR fixed points are studied using the anti-holographic description and the results agree with information available from gauge theory.

Keywords: String Duality, Superstring Vacua, Brane Dynamics in Gauge Theories.

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Contents

1. Introduction

Duality between theories that relates strong and weak coupling is an important tool in the study of strongly coupled systems. The holographic duality, discovered in the study of branes in string/M-theory [1] (for a review, see [2]), is such a duality. It relates the dynamics of a brane system, in a situation in which it decouples from the bulk dynamics, to string/M-theory with the background being the near-horizon geometry of that brane system.

A similar holographic relation was suggested in [3] (and studied further in [4, 5]). The system considered is type-II string theory on

$$R^{d-1,1} \times X^{2n}\,, \tag{1.1}$$

where X^{2n} is a singular Calabi-Yau manifold. In the limit of vanishing string coupling, the only non-trivial dynamics is of modes localized near the singularity, defining a d-dimensional theory without gravity. In [3], an "anti-holographic" formulation of this theory was proposed: type-II string theory in a background of the form

$$\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times \mathcal{N}\,, \tag{1.2}$$

where $\mathbb{R}_\phi$ corresponds to the radial direction (distance from the singularity) and the dilaton varies linearly along this direction. These vacua were first studied in [6] (for a review, see [7]); that they should exhibit holography, was argued in [8].[1] In the backgrounds studied in [3], the manifold X^{2n} in (1.1) is a hypersurface $W(z_1, \ldots, z_{n+1}) = 0$ in flat space $\mathbb{C}^{n+1}$ and it has an isolated singular point. In the present work, this study of the holographic duality is extended to hypersurfaces in a supersymmetry-preserving orbifold $\mathbb{C}^{n+1}/\Gamma$ (where Γ is a finite group of unitary transformations).

For a large class of manifolds X^{2n}, the background (1.1) is T-dual to a configuration with NS5 branes, so the d-dimensional theory describes the decoupled dynamics of these branes. For $d = 6$, the T-duality leads to flat NS5 branes [11, 12, 13] (see also [14]) and the background (1.1) is the near-horizon geometry of these branes [15], $\mathbb{R}_\phi$ being the radial direction. For $d = 4, 2$, one obtains a single NS5 brane wrapped on a manifold Σ [16] and the singular point in X^{2n} transforms to a singular point in Σ. In these cases, the background (1.2) is identified with the near-horizon geometry of this NS5 brane, in the singular region.

The dynamics of a type-IIA NS5 brane wrapped on a Riemann surface Σ is relevant for the study of the low energy dynamics of 4D $N = 2$ supersymmetric gauge theories. This was shown, by a chain of dualities, starting with a background where the gauge theory is seen perturbatively — either in heterotic string theory (as reviewed in [17]) or in brane configurations in type-IIA string theory (as reviewed in [18]). The surface Σ is then identified with the *Seiberg-Witten* (SW) curve [19, 20], which encodes information about the IR dynamics of the gauge theory.

Of particular interest are points in the moduli space, in which the dynamics remains non-trivial in the IR limit. They define interacting 4D $N = 2$ superconformal field theories (SCFT's). These points correspond to singular SW curves. This suggests that such a SCFT is realized in string theory by a type IIA NS5 branes wrapped on a singular surface Σ and, by T-duality, in type-IIB string theory on $R^{3,1} \times X^6$, with a singular X^6. The proposed holographic duality then provides an additional framework in which this SCFT can be studied. One family of such SCFT's was identified [21] in SYM SU(N_c) gauge theory, at points in the moduli space with mutually-non-local massless dyons. These points are indeed realized by singular hypersurfaces in flat space and were considered, using the holographic duality, in [3]. Here this relation is extended to SQCD — SU(N_c) gauge theory with matter ("quarks") in the fundamental representation. It is argued that when there are k massless quarks at the interacting IR SCFT, this theory is realized by a background with a singular hypersurface in a $\mathbb{Z}_k$ orbifold. The orbifold represents the near-horizon region of k KK monopoles, and their frozen 6D gauge symmetry pro-

[1] The relevance of a background with an $\mathbb{R}_\phi$ factor as above to the dynamics at singularities in CY compactification was also suggested, from different points of view, in [9] and [10].

JHEP03(2000)012

vides the global ("flavor") symmetry of the 4D gauge theory. The non-trivial IR SCFT's in the moduli space of SQCD were studied, using SW theory, in [22, 23]. Here these theories are studied using the proposed holographic duality. Specifically, by considering deformations from such a theory, one identifies vector superfields in it and obtains information about their dynamics. All the results agree with what is known from field-theoretic considerations. One family of these SCFT's corresponds to the origin in the moduli space of SU(N_c) gauge theories with $N_f = 2N_c$ massless quarks. These theories are continuously connected to a free theory (by varying the gauge coupling). The implications of this relation are compared to the results obtained from the holographic duality and, again, agreement is found.

The structure of this work is as follows: in the next section, the decoupling limit is reviewed (mostly following [3, 4]), both for singularities and for NS5 branes. The decoupled theory is defined and some general aspects of it are discussed. These aspects are then demonstrated in the case of hypersurface singularities. In section 3, a hypersurface singularity on an orbifold is considered. The holographicly-dual pair is identified and properties of the d-dimensional theory, as obtained from the dual formulations, are discussed. This information is used in section 4, where it is applied to specific cases. In section 4, we concentrate on configurations related to 4D $N = 2$ SQCD. The realization of this theory in string theory is reviewed, the anti-holographic description is identified and used to study the gauge theory. The results of this work are summarized in section 5.

2. The decoupling limit

Consider type-II string theory on

$$R^{d-1,1} \times \bar{X}^{2n} \,,$$

where $\bar{X}^{2n}$ is a compact, n (complex) dimensional Calabi-Yau (CY) manifold, in the limit of vanishing string coupling $g_s \to 0$. When $\bar{X}^{2n}$ is smooth, the dynamics is expected to becomes free in this limit. If, on the other hand, $\bar{X}^{2n}$ has a singular point s, one expects in the above limit a non-trivial dynamics[2] of modes localized near s. If s is "isolated", i.e., has a smooth neighborhood, these modes will decouple from the other modes of the string theory, which are either localized at other singular points or propagate in the bulk of $\bar{X}^{2n}$. In particular, they will decouple from gravity (which corresponds to bulk modes). Therefore, the decoupled dynamics of the modes localized on the singular point s is described by a d-dimensional theory without gravity. This theory is expected to have an anti-holographic description, as explained in the introduction, and it is the subject of the present investigation.

[2]Evidence for such a non-trivial dynamics is provided by the relation between geometrical singularities and NS5 branes, as described below.

JHEP03(2000)012

To obtain a formulation of this theory, it will be useful to have a simpler definition of it. This is indeed possible: one can replace $\bar{X}^{2n}$ by a simpler, non-compact manifold X^{2n}, that shares with $\bar{X}^{2n}$ the same singular region S (i.e., the singular point s and its smooth neighborhood) and the decoupling implies that this change will have no effect on the decoupled dynamics in S. In fact, to define such a d-dimensional theory, one can start directly with a non-compact manifold X^{2n}. This also allows one to consider singularities that cannot be embedded in a compact CY manifold.

A generalization of the above procedure was suggested in [4]: instead of considering the limit $g_s \to 0$ with fixed X^{2n}, one can study a "double-scaling" limit, in which the CY manifold is also modified. For this, one considers a deformation X^{2n}_μ of the above singular manifold X^{2n}, parametrized by a deformation parameter μ. The double-scaling limit is

$$g_s \to 0\,, \qquad \mu \to 0\,, \qquad m = \frac{\mu^r}{g_s} = \text{const.}\,, \tag{2.1}$$

where m is an appropriate ratio that corresponds to some physical quantity in the decoupled theory. The dependence of the limit on this ratio means, in particular, that the separate limits $\mu \to 0$ and $g_s \to 0$ do not commute. The original procedure, with fixed X^{2n}, i.e., first taking $\mu \to 0$ and then $g_s \to 0$, corresponds to $m = 0$, while the opposite order of limits corresponds to $m = \infty$.

There is another limit in which one obtains a theory without gravity which is localized on a singularity: the low energy limit $El_s \to 0$, where l_s is the string length scale and E is the characteristic energy considered. This limit was used, for example, to study the low energy dynamics of 4D $N = 2$ supersymmetric gauge theories (see [17] for reviews). The above two limits are, in fact, two aspects of the same decoupling phenomenon: at fixed finite g_s, one obtains an approximate decoupling of bulk modes below some energy E_d, where $E_d l_s$ depends on g_s (and the moduli of the CY manifold). This becomes an exact decoupling in the extreme IR. As g_s is decreased (with fixed l_s), E_d increases (e.g., the gravitational interaction becomes weaker, since the Planck length $l_p \sim g_s^{1/4} l_s$ decreases) and at vanishing g_s, one obtains a decoupled theory at all energy scales.

The relation to NS5 branes. An important special case is when X^{2n} has a U(1) isometry (acting holomorphicly), and the space Σ of fixed points of this isometry is a (complex) sub-manifold of co-dimension 2. Performing T-duality along the U(1) isometry, one obtains [16] a configuration with an NS5 brane wrapped on Σ. Therefore, in this case, the d-dimensional theory defined above describes the decoupled dynamics of this NS5 brane. The simplest such case is a configuration of parallel KK monopoles. The transverse space (multi Taub-NUT space; $n = 2$) has a U(1) isometry, with fixed points at the location of the monopoles. T-duality along this isometry replaces each KK monopole by an NS5 brane at the same location [11, 12, 13] (see also [14]). When the NS5 branes coincide, one indeed expects, in the $g_s \to 0$ limit, a non-trivial dynamics on these branes [24] (see also [25, 26, 27]). This is because they are dual to decoupled D5 branes, where one obtains, at low energy, a gauge theory with a non-vanishing gauge coupling.

When the KK monopoles are each at a different location, the geometry is smooth everywhere and, therefore, in the decoupling limit, $g_s \to 0$, the dynamics is free. The above T-duality now implies that the decoupled dynamics of a single NS5 brane is free. It is natural to expect that the same remains true when the NS5 brane is wrapped on a smooth manifold. When Σ has a singularity, the same reasoning suggests that the non-trivial dynamics will be confined to the singular region.[3]

The case of a hypersurface singularity. We close this section, by demonstrating some of the issues described above in the specific cases considered in [3, 4, 5] and the present work: singularities embedded in a non-compact hypersurface

$$W(z_1, \ldots, z_{n+1}) = 0$$

in $\mathbb{C}^{n+1}$, or an orbifold of $\mathbb{C}^{n+1}$. The modes localized on the singularity include modes related to D-branes wrapped (fully or partially) on vanishing cycles. These define particles or extended objects in the d-dimensional theory. In the double scaling limit discussed in [4, 5], the quantity m held fixed (see eq. (2.1)) is (proportional to) the mass/tension of these objects (this is explained in subsection 3.1).

When W has the form

$$W = H(z_1, \ldots, z_{n-1}) + uv\,,$$

X^{2n} has a U(1) isometry

$$u \to \alpha u\,, \qquad v \to \frac{v}{\alpha}\,, \qquad |\alpha| = 1$$

and the space Σ of fixed points is[4] $u = v = H = 0$. If the hypersurface is in $\mathbb{C}^{n+1}$, T-duality along the U(1) isometry gives [16] an NS5 brane wrapped on Σ ($H = 0$) in $\mathbb{C}^n$. Similarly, if the hypersurface is in an orbifold $\mathbb{C}^{n-1}/\Gamma \times \mathbb{C}^2_{uv}$, the same T-duality[5] will give an NS5 brane wrapped on Σ ($H = 0$) in $\mathbb{C}^{n-1}/\Gamma \times \mathbb{C}$. In both cases, the singularity of X^{2n} leads to a corresponding singularity of Σ. As argued above, before the T-duality, the non-trivial part of the decoupled dynamics is confined to the singular region of X^{2n}. For the T-dual NS5 brane configuration, this means that

[3] The author is grateful to D. Kutasov for a discussion about this issue.

[4] For $n = 2$, this describes the configuration of parallel KK monopoles discussed above.

[5] In both cases, X^{2n} can be viewed as a fibration uv =const., the base being $\mathbb{C}^{n-1}$ or $\mathbb{C}^{n-1}/\Gamma$, respectively (for $uv \neq 0$, the fiber is $\mathbb{C}^*$). T-duality in the fiber leads to the above NS5 configurations.

JHEP03(2000)012

its decoupled dynamics is localized in the neighborhood of the singularity at Σ, in agreement with the general expectations. The D branes wrapped on vanishing cycles in X^{2n} are dual to D-branes that end (have boundaries) on the NS5 brane in the singular region [16].

3. A singularity on a hypersurface in an orbifold

In this section, we consider the holographic duality (described in the introduction) for singularities modeled by a hypersurface in an orbifold. The holographicly-dual pair is identified in the first subsection. Then, properties of the decoupled theory, as obtained from both dual descriptions, are discussed, first for a general dimension d (in subsection 3.2) and then for $d = 4$ (in subsection 3.3). This serves both to check the proposed duality and as a preparation for the analysis in section 4.

3.1 The holographicly-dual pair

We start by reviewing the case with a hypersurface in flat space, considered in [3, 4, 5], and then introduce the orbifold.

A hypersurface in flat space. A large class of singular manifolds X^{2n} can be modeled by a hypersurface

$$W(z_1, \ldots, z_{n+1}) = 0$$

in $\mathbb{C}^{n+1}$, with W a quasi-homogeneous polynomial, i.e., transforming as $W \to \lambda W$ under

$$z_a \to \lambda^{r_a} z_a \,, \qquad \lambda \in \mathbb{C}\,, \tag{3.1}$$

for some positive (rational) "weights" r_a. The authors of [3] considered such a singularity which has the following additional properties:

- An isolated singularity (point-like; at $z = 0$); the condition for this is that the polynomial W is "transverse"

$$dW = 0 \quad \Leftrightarrow \quad z = 0\,. \tag{3.2}$$

- A singularity that appears at finite distance in the moduli space of (compact, smooth) CY manifolds; as shown in [28], the condition for this is $r_\Omega > 0$, where

$$r_\Omega = \sum_a r_a - 1\,. \tag{3.3}$$

It was proposed in [3] that the dynamics at the singularity in the decoupling limit is described by type-II string theory on

$$\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times \mathrm{U}(1)_Y \times LG_W\,, \tag{3.4}$$

where $\mathbb{R}_\phi \times \mathrm{U}(1)_Y$ is the $N=2$ Liouville theory and LG_W is the $N=2$ Landau-Ginzburg (LG) model with the superpotential being the polynomial W. One way to understand this identification is as follows: to obtain better control on the model, one deforms X^{2n} to a smooth hypersurface X^{2n}_μ by changing W

$$W \to W_\mu = W + \mu V \tag{3.5}$$

(where V is an appropriate quasi-homogeneous polynomial,[6] with weight r_V). Next, this deformed hypersurface is embedded in a weighted projective space: $\mathbb{C}^{n+1}$ is embedded in $\mathbb{WCP}^{n+1}_{-r_\Omega, r_1, \ldots, r_{n+1}}$, by introducing an additional coordinate z_0, with a *negative* weight $-r_\Omega$ (where r_Ω is defined in eq. (3.3)); to embed X^{2n}_μ in this space, W_μ is made quasi-homogeneous, by an appropriate z_0 "dressing":[7]

$$W_\mu \to \hat{W}_\mu = W + \mu \hat{V}\,, \qquad \hat{V} = z_0^{(r_V - 1)/r_\Omega} V\,. \tag{3.6}$$

Now one argues, following [9, 11], that the SUSY NLσM on X^{2n}_μ is described, in the IR limit, by a LG model with the superpotential $\hat{W}_\mu$ (eq. (3.6)).[8][9] Finally, one identifies the z_0 CFT with $N=2$ Liouville theory.[10] This theory has a single chiral superfield $\hat{\phi}$. Denoting the (scalar) bottom component of $\hat{\phi}$ by $\phi + iY$, the dilaton Φ is proportional to ϕ:

$$\Phi = -\frac{Q}{2}\phi\,, \tag{3.7}$$

where Q is a positive parameter. Now, the identification is

$$z_0 = e^{\frac{Q}{2}\hat{\phi}}\,, \qquad Q = \sqrt{2r_\Omega}\,. \tag{3.8}$$

Recall that r_Ω was chosen to be positive (for the singularity to be at finite distance in the CY moduli space), and this is indeed required for the consistency of the above identification. Thus, combining these relations, one obtains the background (3.4).

[6] We assume quasi-homogeneity for simplicity. The arguments below generalize in a straightforward way to a general polynomial, which is of the form $V = \sum_i V_i$, where V_i are quasi-homogeneous. Note that the z_0-dressing in eq. (3.6) will not be the same for all terms.

[7] Note that, to deform the singularity, V must dominate at $z \to 0$. This means that $r_V < 1$, so the power of z_0 in (3.6) is *negative*!

[8] More precisely, this is an orbifold of the above LG model, with a projection on integral U(1) charges of the N=2 superconformal algebra. However, in the present context (of string theory on (3.4)), this orbifold is part of the GSO projection, so it can be ignored at this stage.

[9] In [3], this identification was obtained, for $\mu = 0$, using an embedding in a gauged linear σ model, following [29].

[10] Strictly speaking, this is Liouville theory when $V = 1$, in which case, $\delta\hat{W} = \mu\hat{V} = \mu e^{-\hat{\phi}/Q}$ is the Liouville interaction. This is the case considered in [9, 11], and the z_0 CFT was identified there with the $\mathrm{SL}(2)/\mathrm{U}(1)$ coset SCFT (at level $k = 1/r_\Omega$; see [11] for a review of the evidence for this identification). On the other hand, the analysis in [3] suggested the identification with Liouville theory, as described above. This led to a proposal [4] that these two theories are equivalent.

JHEP03(2000)012

After obtaining the above identification, one may want to remove the deformation μV, returning to the original, singular, hypersurface X^{2n}. However, one finds that the worldsheet formulation of the string theory in the background (3.4) becomes singular. Indeed, the string coupling g_s is related to the dilaton: $g_s = e^{\Phi}$, so eq. (3.7) implies that g_s diverges at $\phi \to -\infty$. For $\mu = 0$, the superpotential is independent of ϕ, so the region $\phi \to -\infty$ is accessible and the perturbative expansion is singular. This singularity is a reflection [10, 11] of the existence of massless solitons: D-branes wrapped on the vanishing cycles at the singularity [30]. Therefore, to obtain a well-behaved perturbative description, it is necessary to make these solitons massive.

This is achieved by the double scaling limit (2.1), suggested in [4, 5]: the deformation discussed there was with $V = 1$ in eq. (3.5) and the quantity held fixed was the ratio

$$m = \frac{\mu^{r_\Omega}}{g_s} \tag{3.9}$$

($r = r_\Omega$ in eq. (2.1)). To understand the meaning of m, one notes that X^{2n}_μ has holomorphic n-cycles that vanish for $\mu = 0$. The volume V_C of such a cycle C is $V_C = \int_C \Omega$, where Ω is the holomorphic n-form on X^{2n}_μ. From the explicit expression for Ω

$$\Omega = \frac{dz_1 \wedge \cdots \wedge dz_{n+1}}{dW}\,, \tag{3.10}$$

one finds that the weight of Ω under (3.1) is r_Ω, defined in eq. (3.3) (and assumed to be positive). This implies that $V_C \sim \mu^{r_\Omega}$ and, therefore, the d-dimensional mass/tension of a D-brane wrapped on C is $V_C/g_s \sim m$. Thus, in the double scaling limit with $m \neq 0$, there are no massless solitons, so one can expect that the worldsheet formulation will be under control and indeed it is: the strong coupling region $\phi \to -\infty$ is inaccessible, because of the deformed superpotential $\hat{W}$ that diverges there. Using the relation $g_s = e^{\Phi}$, one can show that the theory depends on g_s and μ only through the ratio m defined in eq. (3.9) and it can be studied perturbatively, $1/m$ being the string loop expansion parameter.

The orbifold. Considering the above scenario, let Γ be a finite subgroup of $\mathrm{SU}(n+1)$ which commutes with (3.1) (i.e., mixes only coordinates with the same weight) and leaves the polynomial W invariant. The equation $W = 0$ is, therefore, well defined in the orbifold $\mathbb{C}^{n+1}/\Gamma$, and it defines there a hypersurface X^{2n}, with an isolated singularity at $z = 0$. We are interested in the d-dimensional theory describing the decoupled dynamics near this singularity. Note that the singular point $z = 0$ of X^{2n} is at the orbifold singularity, which is what is needed for the decoupled theory to be affected by the orbifold. To obtain an anti-holographic formulation of this theory, we replace the LG_W factor in the background (3.4) by the orbifold LG_W/Γ. Note that Γ defines a symmetry group of LG_W (acting trivially on the supercharges), so one can orbifold by it. We, therefore, propose (generalizing [3]) that the following theories are two formulations of *the same* d-dimensional theory:

- *geometric formulation*: string theory on

$$\mathbb{R}^{d-1,1} \times X^{2n}\,, \tag{3.11}$$

  in the decoupling limit (as described in section 2);

- *worldsheet formulation*: string theory on

$$\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times \mathrm{U}(1)_Y \times LG_W/\Gamma\,. \tag{3.12}$$

3.2 Aspects of the holographic relation

In this and the next subsections, properties of the decoupled d-dimensional theory are identified in the worldsheet formulation. This will be used, in the next section, to analyze specific four-dimensional theories. Some aspects of this identification depend only on the $\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times \mathrm{U}(1)_Y$ factor of the background (3.12), while the factor LG_W/Γ could be replaced by any 2D $(2,2)$ SCFT. These aspects are, therefore, naturally the same here as in the unorbifolded case, discussed in [3]. Other aspects depend also on properties of the LG_W/Γ factor and, accordingly, on the correct identification of the d dimensional theory. As in [3], these aspects are compared to the information available from the geometric formulation and agreement is found, providing evidence for the duality. The results and their implications are summarized in section 5.

Space-time supersymmetry. We start with supersymmetry, showing that in both formulations there are $2^{d/2+1}$ preserved space-time supercharges.

In the worldsheet formulation, the orbifold action in LG_W was chosen to commute with the supercharges and the transformation (3.1) so, in the IR CFT, it commutes with the full $(2,2)$ superconformal algebra (in which one of the U(1) R-symmetries is identified with (3.1)). Therefore, the CFT (3.12) has $(2,2)$ supersymmetry and, as in any such situation, one can construct the space-time supercharges using the U(1) currents of the superconformal algebra. This leads, as before orbifolding (see [3] for more details), to $2^{d/2+1}$ space-time supercharges. The unimodularity of the elements of Γ ($\det(g) = 1$, $\forall g \in \Gamma$) insures that, for an appropriate choice of the orbifold action,[11] the space-time supercharges are Γ-invariant.

In the geometric formulation, before orbifolding there were $2^{d/2+1}$ supercharges (as in any CY compactification), so it remains to check the effect of the orbifold. The supercharges are related to the holomorphic n-form Ω on X^{2n}: indeed, Ω can be written in terms of a covariantly constant spinor η on X^{2n} as $\Omega_{i_1\ldots i_n} = \eta^t \Gamma_{i_1\ldots i_n}\eta$, so the

[11] See [31] for more details (in the notation there, the choice is $(-1)^{K_g} = \det(g)$).

JHEP03(2000)012

orbifold actions on Ω and on the supercharges are related. Using the explicit expression (3.10) for Ω, one finds that Ω is Γ-invariant: both numerator and denominator are invariant (the first invariance follows from the unimodularity of the elements of Γ). Therefore, the orbifold can have at most a $\mathbb{Z}_2$ action on the supercharges. This $\mathbb{Z}_2$-ambiguity has the following meaning: the supercharges appear quadratically in the supersymmetry algebra, so with a given Γ-action on the bosonic coordinates, its action on the supercharges always has a $\mathbb{Z}_2$ ambiguity. The invariance of Ω means that one can choose the Γ-action on the supercharges to be trivial and, with this choice, the orbifold does not break supersymmetry.[12]

Deformations preserving the space-time supersymmetry. Next we discuss SUSY-preserving deformations.

Consider, for example, a deformation of the polynomial W: $\delta W = \mu V$ (where V is a quasi homogeneous polynomial, with weight r_V; as in eq. (3.5)). In the geometric formulation, it induces a change in the manifold X^{2n} and, therefore, also in the corresponding decoupled theory. More specifically, the coefficient (modulus) μ parametrizes a change in the complex structure of X^{2n}, a deformation that does not break supersymmetry. In the worldsheet formulation, a change in W induces a change in the superpotential: $\delta\hat{W} = \mu\hat{V}$, as in eq. (3.6). This means that one adds to the worldsheet lagrangian a top component of a chiral-chiral superfield (of the worldsheet $N = 2$ SCFT). The z_0 dressing insures that this operator has dimension $\Delta = \tilde{\Delta} = 1$. Such a change in the lagrangian preserves the worldsheet superconformal symmetry [32] and, therefore, defines a (truly) marginal perturbation, parametrized by the coefficient (coupling) μ, which preserves the space-time supersymmetry.

This can be generalized for any chiral-chiral primary operator V in the LG_W/Γ factor with equal left and right U(1) charges $f_V = \tilde{f}_V$. V can be "dressed" by the Liouville fields:

$$\hat{V} = e^{\beta(\phi+iY)}V\,. \tag{3.13}$$

This is a chiral-chiral primary of the worldsheet $N = 2$ SCFT, which is the bottom component of a chiral-chiral superfield (we denote both by the same symbol $\hat{V}$). The U(1) charge of $\hat{V}$ (both left and right) is $f_V - \beta Q$ and, to obtain a marginal deformation, as described above, one chooses β to set this charge to 1:

$$\beta = \frac{f_V - 1}{Q}\,. \tag{3.14}$$

This defines a map from the ring of chiral-chiral primary operators (the (c, c) ring) in the LG_W/Γ factor, to SUSY-preserving changes in the decoupled theory.

Before orbifolding, the (c, c) ring in LG_W is spanned by quasi homogeneous polynomials $V(z_a)$, identified modulo $\partial_a W$. Their U(1) charges $f_V, \tilde{f}_V$ are equal and

[12] This choice is the space-time analog of the choice made in the worldsheet formulation (see footnote 11).

coincide with the weight r_V of V under (3.1). In the geometric formulation, the corresponding deformation was identified in [3] with a deformation of the complex structure in X^{2n}, as described above. The orbifold projection truncates this set of deformations in the same way in both the worldsheet and geometric formulations, by restricting the polynomials V to be Γ-invariant. This can be viewed as further evidence for both the holographic relation and the geometric identification of these deformations. Additional (c, c) operators appear in twisted sectors of the orbifold. The corresponding deformations in the d-dimensional theory are related to its influence by the orbifold singularity. This will be seen in specific examples in the next section.

Couplings vs. moduli. In the previous paragraph, possible deformations of the d-dimensional theory were discussed, as viewed in its two formulations. The parameter μ parametrizing such a deformation can be either

- a coupling: parametrizing a change in the theory (e.g., a coefficient of a term in the action); or
- a modulus: parametrizing a change in the vacuum (e.g., a vacuum expectation value (vev)).

In the holographic duality with AdS vacua (for a review, see [2]), considered in the framework of semi-classical supergravity, the distinction between couplings and moduli is related to the behavior of the corresponding bulk fields near the boundary: non normalizable modes are related to couplings in the boundary theory [33, 34] while normalizable modes are related to vevs in the boundary [35, 36] (see also [37]). It is natural to expect the same distinction in the present holographic duality.[13] Here, the "boundary" is at $\phi \to \infty$ and, for a deformation defined by a vertex operator $\hat{V}$, the deformation parameter μ is identified as a coupling iff the corresponding wave function is non-normalizable at $\phi \to \infty$. For the vertex operators (3.13), this implies that for

$$\beta > -\frac{Q}{2}, \tag{3.15}$$

μ is a coupling and for $\beta < -Q/2$, it is a modulus.[14] The case $\beta = -Q/2$ is more delicate and will be discussed below. The bound (3.15) appears also when the string theory on (3.12) is viewed as a 2D CFT coupled to quantum gravity. In that context, (3.15) is the condition for the operator (3.13) to exist as a local operator in the theory (see [38] for more details).

[13] The relation to AdS_{d+1} vacua can be made more concrete in the $d = 2$ case, by using the correspondence, described in [3], between AdS_3 vacua and $\mathbb{R}^{1,1} \times \mathbb{R}_\phi$ vacua.

[14] To obtain these conditions, one notes that the vertex operator $\hat{V}$ is related to the wavefunction Ψ by $\hat{V} = g_s \Psi$, where $g_s = e^{\Phi} = e^{-\frac{Q}{2}\phi}$.

Using eqs. (3.14) and (3.8), one obtains that for

$$f_V > 1 - r_\Omega\,, \tag{3.16}$$

μ is a coupling and for

$$f_V < 1 - r_\Omega\,, \tag{3.17}$$

it is a modulus. This applies, in particular, to the deformations of the polynomial W: $\delta W = \mu V$. For these deformations, the distinction between couplings and moduli was investigated in the geometric formulation in [28] (see also [39]). In this approach, a coupling is a 10D mode whose d-dimensional kinetic energy diverges and, consequently, its fluctuations are "frozen". For $r_V \neq 1 - r_\Omega$, this criterion led to the same conditions on r_V as above, namely, the conditions (3.16), (3.17) with $f_V = r_V$. The deformation with $r_V = 1 - r_\Omega$ is found to be a coupling. It is interesting to note that also in this approach, μ was identified as a non-fluctuating coupling when the perturbation was supported (in some sense) far from the singularity (which is indeed identified here with $\phi \to \infty$). The manifold considered in [28] was a hypersurface in *flat space*. However, the aspects analyzed were convergence of certain integrals, and these aspects are not influenced by a finite group of identifications. Therefore, the results of [28] apply also to the present case.

R-symmetry. In the worldsheet formulation, the scalar Y is free, and this leads to two U(1) symmetries, with the following conserved charges

$$R = i\frac{2}{Q}\oint \partial Y\,, \qquad \tilde{R} = i\frac{2}{Q}\oint \bar{\partial} Y\,. \tag{3.18}$$

The space-time supercharges are charged under these symmetries, (see [3] for details), so these are R-symmetries. Defining the linear combinations

$$R_\pm = R \pm \tilde{R}\,, \tag{3.19}$$

all supercharges have charge $|R_\pm| = 1$. For the deformations defined by $\hat{V}$ in eq. (3.13), $R = \tilde{R}$, so

$$R_-(\mu) = 0\,, \qquad R_+(\mu) = -R_+(\hat{V}) = -2R(\hat{V}) = -\frac{4\beta}{Q} = 2\frac{1-f_V}{r_\Omega} \tag{3.20}$$

(using eqs. (3.13), (3.14) and (3.8) and the fact that the worldsheet action is invariant). Combining eqs. (3.16), (3.17) and (3.20), one obtains that μ is a coupling when $R_+(\mu) < 2$ and a modulus when $R_+(\mu) > 2$.

In the geometric formulation, the transformation (3.1) with $|\lambda| = 1$ is an isometry of X^{2n} and, therefore, induces a U(1) symmetry in the corresponding theory. To identify its action on supercharges, one considers again the holomorphic n-form Ω. From eq. (3.10) one finds that the weight of Ω under (3.1) is r_Ω, defined in eq. (3.3), and the fact that it is non-zero means that the above U(1) symmetry is an R-symmetry. With a normalization in which the supercharges have U(1) charge ± 1, the R-charge R'_+ is related to the weight r by

$$R'_+ = 2\frac{r}{r_\Omega}\,.$$

It was suggested in [3] that R'_+ should be identified with R_+. This was checked by verifying that, for deformations of W: $\delta W = \mu V$ (where $f_V = r_V$), the geometric formulation indeed gives the same R-charge as obtained in eq. (3.20):

$$R'_+(\mu) = 2\frac{r_\mu}{r_\Omega} = 2\frac{1 - r_V}{r_\Omega} \tag{3.21}$$

(where the last equality follows from the fact that the polynomial W has weight $r_W = 1$).

3.3 Four-dimensional theories

We move now to a closer look at the case $d = 4$. The number of supercharges is 8, corresponding to $N = 2$ supersymmetry in four dimensions.

Coupling-moduli pairing. The (c, c) ring of the LG_W/Γ CFT has a $\mathbb{Z}_2$ "reflection" symmetry, relating operators with U(1) charges f and $\hat{c} - f$, where $3\hat{c}$ is the central charge of the CFT:

$$\hat{c} = \sum_a (1 - 2r_a) = (n - 1) - 2r_\Omega\,. \tag{3.22}$$

To identify this symmetry, one uses two bijective relations between the (c, c) ring and the (a, a) ring — the ring of antichiral-antichiral primary operators. One relation is charge conjugation and the other is obtained using the spectral flow of the $N = 2$ superconformal algebra (the existence of this second relation is a consequence of the unimodularity of the elements of Γ [31]). Combining these two, one obtains a bijection in the (c, c) ring. As to the U(1) charge, starting with a (c, c) operator with charge f, complex conjugation gives an (a, a) operator with charge $-f$ and then the spectral flow, gives a (c, c) operator with charge $\hat{c} - f$.

This reflection symmetry is not specific to models related to four-dimensional singularities. However, its significance is enhanced for $d = 4$, as we now explain. In this case, $\hat{c} = 2(1 - r_\Omega)$ (using eq. (3.22) with $n = 3$). Comparing to the bounds (3.16) and (3.17), one can see that, for operators with $f \neq \hat{c}/2$, the reflection symmetry induces a pairing between coupling deformations and moduli deformations. This pairing has a natural interpretation in the 4D theory (as was also observed independently in [39]). The deformations defined by (c, c) operators are related, in the 4D theory to scalars in vector superfields. When the deformation parameter μ is a coupling with $R_+(\mu) < 2$, it corresponds to adding a *top component* $\mathcal{A}_t$ of a vector superfield $\mathcal{A}$ to the prepotential (with μ as a coefficient), while if it is a modulus with $R_+(\mu) > 2$, the change is in the vev of a *bottom component* $\mathcal{A}_b$ of a vector superfield $\mathcal{A}$. Thus, each vector superfield $\mathcal{A}$ in the 4D theory defines two deformations - one coupling and one modulus — and it is natural to identify these pairs with the pairs seen in the worldsheet formulation.

Evidence for this identification is obtained by considering R-charges. The 4D $N = 2$ supersymmetry algebra has a U(1)×SU(2) R symmetry group. In the previous subsections, a $U(1)_+ \times U(1)_-$ R-symmetry group was found (with charges (3.18). To identify the relation between these two groups, one notes that the scalars in vector superfields are neutral under the SU(2) factor of the R-symmetry and charged under the U(1) factor. The charges found in subsection 3.2 (eq. (3.20)) imply that R_+ should be identified with the U(1) factor and R_- with a U(1) subgroup of the SU(2) factor. Now, considering a pair μ_c, μ_m of deformation parameters, related by the reflection symmetry described above, one finds in both approaches that the sum of their R-charges is four, providing evidence for their correspondence to the same vector superfield $\mathcal{A}$. In the worldsheet formulation this follows from eq. (3.20), while in the 4D field theory this follows from:[15]

$$R_+(\mu_m) = R_+(\mathcal{A}_b)\,, \qquad R_+(\mu_c) = -R_+(\mathcal{A}_t) = -[R_+(\mathcal{A}_b) - 4]\,. \tag{3.23}$$

IR conformal dimensions and unitarity. In the extreme IR limit, one obtains an $N = 2$ superconformal field theory. The 4D $N = 2$ superconformal algebra (SCA) includes a U(1)×SU(2) R-symmetry and implies a relation between the R-symmetry quantum numbers of an operator and its conformal dimension [40]. In particular, the bottom component of a vector superfield is a chiral primary field of the SCA, a scalar of the SU(2) R-symmetry and a Lorentz scalar. For such a field, the conformal dimension is $D = \frac{1}{2}R''_+$, where R''_+ is the charge of the U(1) R-symmetry. Identifying R''_+ with the charge R_+ (3.19),[16] one can determine the conformal dimensions of the 4D vector superfield related to a deformation defined by a worldsheet (c, c) operator V: if $\mu = \mu_c$ is a coupling with $R_+(\mu) < 2$,

$$D(\mathcal{A}_b) = \frac{1}{2}R_+(\mathcal{A}_b) = \frac{1}{2}[R_+(\mathcal{A}_t) + 4] = \frac{1}{2}[-R_+(\mu_c) + 4] = \frac{f_V - 1}{r_\Omega} + 2 \tag{3.24}$$

and if $\mu = \mu_m$ is a modulus with $R_+(\mu) > 2$,

$$D(\mathcal{A}_b) = \frac{1}{2}R_+(\mathcal{A}_b) = \frac{1}{2}R_+(\mu_m) = \frac{1 - f_V}{r_\Omega}\,, \tag{3.25}$$

(where, in the last stage, we used (3.20)).

[15] The U(1) factor of the R-symmetry is defined only up to a shift by a U(1) symmetry that commutes with the supercharges. However, the relations (3.23) are invariant under such a shift, so they indeed provide evidence for the proposed interpretation of the pair of deformations.

[16] This identification will be discussed further at the end of this section.

JHEP03(2000)012

As observed in [3], the bound (3.16) for deformations that are couplings gives, when substituted in (3.24)), $D(\mathcal{A}_b) > 1$, which is (almost; see below) the unitarity bound on the conformal dimension of $\mathcal{A}_b$. The same is true for deformations that are moduli: the bound is (3.17) and when it is substituted in (3.25)), it also gives[17] $D(\mathcal{A}_b) > 1$.

Deformations with $R_+(\mu) = 2$. So far, we only discussed the identification, in the 4D theory, of deformations with $R_+(\mu) \neq 2$. We turn now to those with $R_+(\mu) = 2$. The precise unitarity bound is $D(\mathcal{A}_b) \geq 1$ so, for $R_+(\mu) = 2$, it allows both possibilities — a coupling and a modulus. In either case, this leads to the identification of a vector superfield $\mathcal{A}$ with $D(\mathcal{A}_b) = 1$. The SCA implies that such a superfield is free. The geometric formulation suggests [28] that μ is a coupling (as described in the previous subsection) so its natural interpretation would be as that for $R_+(\mu) < 2$: a coefficient of a term $\mathcal{A}$ in the superpotential. However, as observed in [22] (using $D(\mathcal{A}) = 1$ and the SCA), the corresponding contribution to the lagrangian is a total derivative and, therefore, has no effect. Instead, we propose, following [22], that in this case, μ corresponds to a vev (of a bottom component) of a vector superfield $\mathcal{A}$ with the following properties:

- $\mathcal{A}$ is *frozen* in the IR limit, i.e., its kinetic energy diverges in this limit;
- $\mathcal{A}$ couples to a conserved current which is non-trivial in the IR SCFT, i.e., the corresponding symmetry, which becomes a global symmetry in the IR, acts non-trivially in this SCFT.

The simplest realization of such a situation is $N = 2$ SQED: a single vector multiplet (leading to a U(1) gauge theory) with a charged massless hypermultiplet. Because of the non-trivial charge, the gauge coupling vanishes in the IR limit, leading to a divergent kinetic energy.

According to the above proposal, μ is a coupling in the IR SCFT, in agreement with the analysis in the geometric formulation. However, it originates from a modulus. Moreover, it may become a modulus also in the IR, upon deformation: for example, in SQED, deforming the theory by giving a vev μ to (the bottom component of) the vector superfield leads to a massive hypermultiplet. In the deformed theory, the gauge coupling does not vanish in the IR and, consequently μ remains a vev of a fluctuating (although free) field. This mixed nature of μ is, in fact, suggested also by the ϕ dressing in the worldsheet formulation, as described in the previous subsection. For $\beta > -Q/2$, the wave function is exponentially supported at $\phi \to \infty$ and vanishes at $\phi \to -\infty$, while for $\beta < -Q/2$, the situation is reversed. The present case corresponds to $\beta = -Q/2$ and it is special in having support at both regions.

[17] This was observed independently also in [39].

The above identification also leads to the correct conformal dimension for μ. This can be argued as follows [22]: since a conserved charge is dimensionless, the corresponding conserved current has dimension three, so the vector field coupling to it has dimension one; the SCA now implies that the bottom component of the corresponding superfield has also dimension one. We end with some comments:

- The dimension of the parameter μ is, by definition, $D(\mu) = D(\mathcal{A}_b)$ if it is a modulus; and $D(\mu) = 4 - D(\mathcal{A}_t)$ if it is a coupling. In both cases, this gives

$$D(\mu) = \frac{1}{2} R_+(\mu)\,, \tag{3.26}$$

therefore,

$$\mu \text{ is a coupling} \quad \Longleftrightarrow \quad D(\mu) \leq 1\ .$$

- As explained above, a free vector superfield $\mathcal{A}$ (with $D(\mathcal{A}) = 1$) leads only to a single deformation of the theory. In the worldsheet formulation this means that one should not expect to identify, in the LG_W/Γ CFT, a pairing between (c,c) operators with charge $f = \hat{c}$. Indeed, one finds that the number of such operators is not always even.

- It was suggested in [22] that a coupling μ with $D(\mu) < 1$ can also be identified with a vev of a vector superfield $\mathcal{B}$ that, in the IR limit, is frozen but dues not couple to a non-trivial conserved current. Such a superfield would couple to the SCFT through a term $\mathcal{A}\mathcal{B}/\Lambda^\delta$ in the prepotential, where $\mathcal{A}$ is an interacting vector superfield in the SCFT and Λ is some scale in the underlying theory (note that $\delta = D(\mathcal{A}) - 1 > 0$, so this is an irrelevant interaction). The parameter μ is then identified as $\mu = \langle \mathcal{B}_b \rangle / \Lambda^\delta$. With this identification, all the deformations are identified with vevs of vector superfields: interacting for $D(\mu) > 1$ and frozen for $D(\mu) \leq 0$.

Relevance. There is another property of the deformation that can be deduced from the dimension D of the deformation parameter μ: relevance. Marginal deformations correspond to $D(\mu) = 0$, relevant deformations — to $D(\mu) > 0$ and irrelevant — to $D(\mu) < 0$. This distinction can also be identified in the worldsheet formulation:[18] as described in the introduction, the coordinate ϕ parametrizes the distance from the singularity. As in the holographic dualities with AdS vacua, radial motion in the near-horizon geometry corresponds, in the decoupled theory, to RG flow [41] (see

[18]In the geometric formulation, for deformations $\delta W = \mu V$, relevance means dominance of V at $z \to 0$: for a relevant deformation ($D(\mu) > 0$), $r_V < 1$ (see eq. (3.21)), so V dominates over W at $z \to 0$ and makes a macroscopic change in the singularity; for a marginal deformation ($D(\mu) = 0$), μ parametrizes a continuous change of the singularity; and for an irrelevant deformation, V is negligible at $z \to 0$ and has no effect on the singularity.

JHEP03(2000)012

also [42, 43]): large distances ($\phi \to \infty$) correspond to high energies (UV) and small distances ($\phi \to -\infty$) — to low energies (UV). Therefore, a vertex operator (3.13) with $\beta < 0$ defines a perturbation that increases in the IR — a relevant perturbation. Similarly, $\beta < 0$ corresponds to an irrelevant perturbation and $\beta = 0$, to a marginal one. Since

$$D(\mu) = -\frac{2\beta}{Q}$$

(see eqs. (3.26) and (3.20)), these two approaches to identify relevance give the same result.

To summarize, we have made the identification $R''_+ = R_+$ and it was used to reproduce, using the worldsheet formulation, two bounds in the decoupled IR theory: the unitarity bound and relevance. The supercharges indeed satisfy $R''_+ = R_+$, so the non-trivial content of this identification is the charge of the vector superfields (which are the only superfields discussed in this work). Any one of the bounds could be used to *deduce* this equality. Then, the agreement in all other aspects, as described above, serves as additional evidence for the proposed holographic relation.

4. IR fixed points in 4D $N = 2$ SUSY gauge theories

In this section, we restrict attention to configurations relevant for the study of 4D $N = 2$ supersymmetric gauge theories. Specifically, we consider SQCD: SU(N_c) gauge theory with N_f hypermultiplets ("quarks") in the fundamental representation of the gauge group.

4.1 Realization of SQCD in string theory

4D $N = 2$ SQCD can be realized in string theory as follows (for a review, see [18]): one considers type-IIA string theory with the following brane configuration: all branes are extended in the (0123) direction; there are two NS5 branes which are also extended in (45), and N_f D6 branes which are also extended in (789); finally, there are D4 branes extended also in the (6) direction over finite intervals, ending on the other branes. When there are N_c coinciding D4 branes extending between the two NS5 branes, the low-energy dynamics of the D4 branes is described by 4D $N = 2$ SQCD, as defined above [44]. The vector multiplets correspond to open strings between the D4 branes, and the hypermultiplets correspond to open strings between the D4 branes and the D6 branes. The dynamics of the NS5 and D6 branes is considered "frozen" in the 4D theory, since these branes have infinite extension in the "internal" directions. In particular, the U(k) global ("flavor") symmetry is the frozen gauge symmetry of the D6 branes. The relative location of these branes determines the parameters of the gauge theory. In particular, the (45) locations of the D6 branes determine the mass parameters of the hypermultiplets. In a given NS5-D6 configuration, the possible locations of the D4 branes (consistent with supersymmetry) parametrize the moduli space of the gauge theory. In particular, distributing the D4 branes in the (45) directions (along the NS5 branes), corresponds to the Coulomb branch, parametrized by vacuum expectation values for the scalars in the vector multiplets.

Lifting this configuration to M-theory [45], the D6 branes are identified as KK monopoles, corresponding to a non-trivial 4D transverse space $\mathcal{M}$ (including the directions (456)) — the multi Taub-NUT space [46]. One of the complex structures of $\mathcal{M}$ is that of a hypersurface

$$zw = Q(x) \tag{4.1}$$

in $\mathbb{C}^3$, where $Q(x)$ is a polynomial of degree N_f with the coefficients related to the mass parameters of the hypermultiplets. In configurations corresponding to the Coulomb branch of the gauge theory, the NS5 and D4 branes combine to a single M5 brane, wrapped on a 2D Riemann surface Σ, embedded holomorphicly in $\mathcal{M}$. This surface is identified with the *Seiberg-Witten (SW) curve* of the gauge theory [19, 20].[19] In the representation (4.1) of $\mathcal{M}$, Σ is the curve $H = 0$, where[20]

$$H = z + gw - 2P(x)\,. \tag{4.2}$$

Here g is a function of the gauge coupling and $P(x)$ is a polynomial of degree N_c, with the coefficients being the moduli of the Coulomb branch.

Compactifying the (7) direction, one obtains back type-IIA string theory, this time with N_f KK monopoles — corresponding to the space $\mathbb{R}^{3,1} \times \mathbb{C} \times \mathcal{M}$ — and an NS5 brane extended in $\mathbb{R}^{3,1}$ and wrapped on Σ. This is a situation of the type considered in the introduction.

Naively, the above relations seem to suggest that the decoupled dynamics on this NS5 brane is described by $N = 2$ SQCD. This is not quite so, because the second description is valid in a range of parameters (of string theory) which is different from that in which the gauge theory was identified in the first configuration. However, there is evidence that some aspects of the low energy dynamics ,including those studied below, are not sensitive to the above changes and, therefore, are shared by the gauge theory and the dynamics of the NS5 brane. Moreover, there is another chain of dualities relating gauge theories (realized this time in heterotic string theory) and the dynamics of NS5 branes in the same sense as above (see [17] for reviews).

4.2 Interacting IR fixed points in the moduli space

In most of the vacua in the Coulomb branch of $N = 2$ SQCD, the massless fields are vector superfields corresponding to an abelian gauge symmetry and, possibly,

[19] This curve was determined, using field-theoretic considerations, in [19, 47, 48] (for pure SU(N_c) SYM) and [20, 49, 50] (for SQCD).

[20] Substituting $w = Q(x)/z$ and $z = y + P(x)$ in $H = 0$, one obtains the familiar form

$$y^2 = P(x)^2 - gQ(x)\,.$$

JHEP03(2000)012

additional electrically-charged hypermultiplets. In these vacua, the dynamics is free in the IR. However, there are vacua with additional massless fields, for which the IR dynamics is non-trivial, defining an interacting superconformal field theory (SCFT). This is the case for SQCD with $N_f = 2N_c$ massless quarks, at the origin of the Coulomb branch, where the additional massless fields can be identified, at weak coupling, as vector superfields, enhancing the gauge symmetry to a non-abelian group. Other non-trivial SCFT's are obtained at points in the Coulomb branch in which mutually non-local hypermultiplets become massless [21]

All these vacua correspond to a SW curve Σ with an isolated singularity so, in the stringy realization, there is an NS5 brane wrapped on a surface Σ with an isolated singularity. As explained in section 2, the only part of the configuration that is relevant for the decoupled dynamics on the NS5 brane is the neighborhood of the singular point. This implies that the relevant KK monopoles are those that are *at* the singularity. With the singular point in Σ being at $x = 0$, $Q(x)$ in (4.1) can be represented by $Q(x) = x^k$ ($k \leq N_f$), corresponding to k coinciding monopoles; in the gauge theory this corresponds to k massless quarks. For the same reason, only the region near the center of the KK monopoles is relevant and there the geometry is that of an orbifold $\mathbb{C}^2/\mathbb{Z}_k$. This orbifold can be parametrized by two flat coordinates z', w', subject to the identification

$$z' \to \alpha z' \,, \qquad w' \to \frac{w'}{\alpha} \,, \qquad \alpha = e^{2\pi i/k} \,, \tag{4.3}$$

and the coordinates z, w, x in (4.1) are $\mathbb{Z}_k$-invariant functions of z', w':

$$z = z'^k \,, \qquad w = w'^k \,, \qquad x = z'w' \,. \tag{4.4}$$

To summarize, we can consider type-IIA string theory on $\mathbb{R}^{3,1} \times \mathbb{C}^2_{z'w'}/\mathbb{Z}_k \times \mathbb{C}$ with an NS5 brane on $\mathbb{R}^4 \times \Sigma$, where Σ is the 2D surface $H(z, w, x) = 0$ in $\mathbb{C}^2_{z'w'}/\mathbb{Z}_k$. This has, as a dual description (see footnote 5), type-IIB string theory on $\mathbb{R}^{3,1} \times X^6$, where X^6 is a hypersurface $W = 0$ in $\mathbb{C}^2_{z'w'}/\mathbb{Z}_k \times \mathbb{C}^2_{uv}$, with the $\mathbb{Z}_k$ action (4.3) and the polynomial

$$W = H(z, w, x) + uv \,. \tag{4.5}$$

The surface Σ ($H = 0$) has an isolated singularity at the origin and, therefore, so does X^6 ($W = 0$). For a quasi-homogeneous H, this looks like a configuration of the type considered in the previous section (with $d = 4$, $n = 3$ and $\Gamma = \mathbb{Z}_k$). Actually, there is a slight difference in the configurations but, as we shall argue below, it is irrelevant for the study of the 4D gauge theory. The difference is the following: the orbifold considered in the previous section has a non-singular worldsheet description and leads here to a $\mathrm{U}(1)^k$ 6D gauge symmetry. In the present construction, one has in mind an orbifold with an enhanced $\mathrm{U}(k)$ gauge symmetry, the extra massless states being related (in the type-IIA picture) to D2 branes wrapped on the vanishing cycles. The difference in the backgrounds is that in the first one there is a non-vanishing B-field, leading to non-vanishing masses for the above wrapped D2 branes [51]. However, in the present context, i.e., concentrating on the 4D dynamics, the 6D gauge symmetry on the orbifold is considered frozen anyway, therefore, it is reasonable to expect that the above difference in the 6D dynamics has no effect on the 4D theory. Therefore, one can identify the present configuration as one of those considered in the previous section.

Study of the interacting SCFT's. Using the above identification, one can use the anti-holographic description proposed in section 3 to obtain information about the interacting 4D SCFT's. The general idea is to study deformations of the theory and through them, to identify operators in it and to obtain information about their dynamics. We concentrate, as in section 3, on SUSY-preserving deformations, defined in the worldsheet formulation by (c, c) operators of the $LG_W/\mathbb{Z}_k$ CFT. As described in subsection 3.3, these are related to vector superfields in the 4D theory and this relation depends on the dimension $D = \frac{1}{2}R_+$ of the corresponding deformation parameter μ, as summarized below:

- $D(\mu) > 1$: $\mu \sim \langle \mathcal{A}_b \rangle$, where $\mathcal{A}_b$ is the bottom component of a superfield $\mathcal{A}$, whose dimension is $D(\mathcal{A}_b) = D(\mu)$; the dimension $D > 1$ indicates that this superfield is involved in a non-trivial interaction;
- $D(\mu) = 1$: $\mu \sim \langle \mathcal{B}_b \rangle$, where $\mathcal{B}_b$ is the bottom component of a superfield $\mathcal{B}$ which, in the IR, is frozen but couples to a non-trivial conserved current;
- $D(\mu) < 1$: the deformation is a change $\sim \mu\mathcal{A}$ in the prepotential of the 4D SCFT, where $\mathcal{A}$ is an interacting vector superfield of dimension $D(\mathcal{A}) = 2 - D(\mu) > 1$; μ is, possibly, related to a vev $\mu \sim \langle \mathcal{B} \rangle / \Lambda^\delta$ of a free superfield $\mathcal{B}$, which does not couple to a conserved current in the IR SCFT and its only interaction with this theory is through an irrelevant term $\mathcal{A}\mathcal{B}/\Lambda^\delta$ in the superpotential (where Λ is a scale in the underlying 4D theory and $\delta = 1 - D(\mu) > 0$).

The deformations with $D(\mu) \neq 1$ are expected to appear in pairs (μ_m, μ_c), each pair corresponding to an interacting vector superfield $\mathcal{A}$, with the conformal dimensions related by

$$D(\mathcal{A}) = D(\mu_m) = 2 - D(\mu_c)\,. \tag{4.6}$$

Some of this analysis can be performed directly in the gauge theory. Indeed, for a deformation of the polynomial W, the dimension of the deformation parameter can be obtained from eq. (3.21):

$$D(\mu) = \frac{1}{2}R'_+(\mu) = \frac{r_\mu}{r_\Omega} = \frac{1 - r_V}{r_\Omega}\,, \tag{4.7}$$

thus using only geometric information about X^6: the polynomial W and the R-charge of the holomorphic 3-form Ω. This information can be translated to purely gauge-theoretical data: the deformations of W (4.5) are the deformations of the SW curve and the SW differential can be expressed as an integral over the holomorphic 3-form Ω [16], so they have the same R-charge. Therefore, one can use the SW theory to find dimensions of deformation parameters. This was indeed done in [21, 22, 23]. However, the parametrization of the SW curve is not unique: for example, the polynomials $P(x)$ and $Q(x)$ could be parametrized either by their coefficients or by their roots, leading to a different set of conformal dimensions. The embedding in string theory and the explicit worldsheet formulation fixes this ambiguity (by identifying the deformation parameters with coefficients of terms in the worldsheet lagrangian).[21] This demonstrates the usefulness of the holographic relation for the study of these IR SCFT's.

We now analyze several families of SCFT's, using the worldsheet formulation and compare the results to information from field theory. Note that, in all cases, the superfields u, v in eq. (4.5) decouple (they are massive), so $LG_W = LG_H$.

4.3 Singular points in SYM

For a singularity away from KK monopoles, one can set $Q = 1$ in eq. (4.1). These are the singularities appearing in pure $N = 2$ SU(N_c) SYM theories.[22]

In this case, g in eq. (4.2) can be rescaled to 1 (by rescaling the coordinates z, w, x and the parameters in $P(x)$). This is a reflection of the fact that the gauge coupling in the pure SYM theory is transmuted to a scale and is not a real parameter in the theory. The singularity appears for $P - 1 \sim x^l$ (with $2 \leq l \leq N_c$). The singular point is at $z = w = 1$, and expanding around it (with $z = 1 + \tilde{z}$), one obtains

$$H \sim \tilde{z}^2 - 2(P-1) \sim \tilde{z}^2 - 2x^l \,. \tag{4.8}$$

In the worldsheet formulation, the LG_H CFT is a minimal model. The (c, c) ring is spanned by

$$V = x^{l-j}\,, \qquad j = 2, \ldots, l\,.$$

Applying eq. (3.3) to (4.5) and (4.8), one obtains $r_\Omega = 1/2 + 1/l$, which gives [23, 3], for the spectrum of conformal dimensions (using eq. (4.7)):

$$r_\mu = j r_x = \frac{j}{l} \quad \to \quad D(\mu) = \frac{2j}{l+2}\,.$$

This spectrum corresponds to $[\frac{1}{2}(l-1)]$ interacting vector superfields (where [...] denotes the integer part):

$$\mathcal{U}_j\,, \qquad D = \frac{2j}{l+2}\,, \qquad j = \left[\frac{l}{2} + 2\right], \ldots, l \tag{4.9}$$

and, for even l, an additional IR-free vector superfield (with $D = 1$).

[21] As will be seen below, for the polynomials $P(x), Q(x)$, the correct parameters are the coefficients of $P(x)$ and the roots of $Q(x)$.

[22] Unlike the singularities discussed in the next subsection, these singularities appear only in the strong coupling region of the Coulomb moduli space and do not have a semi-classical interpretation.

Choosing $l = N_c$ (i.e., the most singular point for a given N_c), one finds a nice agreement with the SW theory: this singularity occurs at a singe point in the moduli space, which is $(l-1)$-dimensional, so one expects to find $l-1$ different relevant deformations of the IR SCFT. Indeed, the present approach leads to $l-1$ deformations and all of them are relevant: either vevs or relevant couplings ($D(\mu) < 0$). The SW theory also provides the charges (under the unbroken $\mathrm{U}(1)^{l-1}$ gauge group) of the dyons that become massless at the singular point [19]. In the present case one finds that (for an appropriate choice of duality frame) out of the $l-1$ gauge fields, $[\frac{1}{2}(l-1)]$ couple to mutually-non-local dyons and, for even l, there is one more gauge field with an electrically-charged hypermultiplet. This is in full agreement with the present results: the interacting superfields $\mathcal{U}_j$ are those coupled to mutually-non-local dyons and the free superfield with $D = 1$ is the one coupled only electrically.

The case $l = 2$ is special, since it corresponds to a free IR SCFT. Indeed, this singularity indicates in the gauge theory the appearance of a massless dyon. The low energy theory is SQED: a U(1) vector multiplet with an electrically charged hypermultiplet.[23] There is a single deformation of this theory: a change of the vev of the above vector superfield, giving a mass to the dyon. Because of the electric charge, the U(1) gauge coupling flows to zero in the IR, so the vector superfield is frozen in the IR and, correspondingly, the deformation parameter has $D = 1$. This is exactly what was found in the worldsheet formulation.

4.4 Singular points in SQCD

We turn to singularities with a non-trivial flavor symmetry ($k \geq 2$). The singular point is on the orbifold singularity $z' = w' = 0$ and, neglecting sub-leading terms, one obtains

$$H = z + gw + x^l = z'^k + gw'^k - 2h(z'w')^l \tag{4.10}$$

(corresponding to $P(x) \sim hx^l$). We now distinguish between three situations:

- $k < 2l$: In this case the last term is sub-leading, so H in eq. (4.10) becomes

$$H = z'^k + gw'^k \, . \tag{4.11}$$

  Note that this is independent of l, therefore, so is the physics at this singularity. The LG_H CFT is a product of two decoupled minimal models (they become coupled by the $\mathbb{Z}_k$ orbifold action). In the untwisted sector, the (c, c) ring is spanned by $\mathbb{Z}_k$-invariant polynomials in z', w' (which is the same as arbitrary polynomials in z, w, x), with identifications modulo $\partial_{z'}H, \partial_{w'}H$. A possible choice for a basis is

$$V_j = x^j \, , \qquad j = 0, \dots k-2 \, .$$

[23] This vector multiplet is related to the UV vector multiplet by a duality transformation [19].

Applying eq. (3.3) to (4.5) and (4.11), one obtains $r_\Omega = 2/k = r_x$, which gives [23], for the spectrum of conformal dimensions (using eq. (4.7)):

$$r_\mu = \left(\frac{k}{2} - j\right) r_x \;\to\; D(\mu) = \frac{k}{2} - j\,.$$

The twisted sector of a LG orbifold was analyzed in [31]. Applying this analysis to the present case, one finds in each sector one state V_i' ($i = 1, \ldots, 2N-1$) with a U(1) charge $f_{V_i'} = 1 - r_\Omega$, corresponding to $D(\mu) = 1$. This leads to the identification of $[\frac{1}{2}(k-2)]$ vector superfields (where $[\ldots]$ denotes the integer part)

$$\mathcal{U}_D\,, \qquad \begin{cases} D = 2, 3, \ldots, \frac{k}{2},\ k \text{ even} \\ D = \frac{3}{2}, \frac{5}{2}, \ldots \frac{k}{2},\ k \text{ odd} \end{cases} \tag{4.12}$$

and $2[k/2]$ vector superfields with $D = 1$.

- $k = 2l$: In this case, H in eq. (4.10) is quasi homogeneous and, for $g \neq h$, the surface $H = 0$ has an isolated singularity (the case $g = h$ will be discussed below). As in the previous case, the LG_H CFT is a product of two minimal models, the only difference being that they interact through the last term in H. This difference, however, has no effect on the chiral ring, which is, therefore, the same as for $k < 2l$.

- $k > 2l$: In this case, H is not quasi-homogeneous. Naively, at least one of the first two terms is negligible (i.e., has a higher weight), however, neglecting it would lead to a non-isolated singularity (at $z' = 0$ or $w' = 0$). A closer look reveals that the first term is leading in the direction $w' = 0$ and the second, in the direction $z' = 0$, so the precise statement is that this kind of singularity cannot be described by a quasi-homogeneous polynomial and, therefore, does not have a worldsheet description of the form discussed in section 3. Note, however, that it can be realized indirectly, as a deformation of $k \leq 2l$ singularities

Comparing to field theory. Each of the above singularities appears in SQCD with $l \leq N_c$, $k \leq N_f$, where k of the quarks have the same mass parameter. Moreover, each such singularity can be found in the *semi-classical region* of an asymptotically-free theory (with $N_f < 2N_c$ and $l < N_c$), i.e., with mass parameters and vevs much larger then the strong coupling scale. There, using semi-classical considerations, it can be identified as the IR limit of SU(l) gauge theory with k massless quarks, at the origin of the moduli space.

Using this identification, one can demonstrate that the holographic duality considered here is a strong-weak coupling duality, in the sense that there is no situation in which both the gauge theory and the worldsheet CFT are weakly coupled. For

$k < 2l$, the LG_H theory is a product of two decoupled minimal models, so the worldsheet CFT is solvable, but the gauge theory is asymptotically free and strongly coupled in the IR. For $k > 2l$, the gauge theory is free in the IR but the LG_H CFT is complicated. Finally, for $k = 2l$ the situation depends on the interaction strength of the last term in H (in eq. (4.10)). It is proportional to $h/\sqrt{g}$ (as can be seen by rescaling $w' \to w'/g^{1/k}$), so the minimal models are decoupled for $h/\sqrt{g} \to 0$ while the gauge theory is free in the opposite limit (as will be described below).

As noted above, the singularity with $k < 2l$ is independent of l. Moreover, for even k it is the same as $k = 2l$ with $h \to 0$. This last relation can be understood also in field theory, considering SQCD with $N_f = k < 2l < 2N_c$ with even k, where the $\mathrm{SU}(N_c)$ gauge group is broken to $\mathrm{SU}(l)$ at some scale M and further to $\mathrm{SU}(l')$ ($l' = \frac{1}{2}k$) at a lower scale[24] M'. This gives the (k, l') singularity, with

$$h \sim \left(M^{N_c - l} M'^{l - l'}\right)^2 .$$

Unbroken $\mathrm{SU}(l)$ gauge group (corresponding to the (k, l) singularity) is obtained in the limit $M' \to 0$, which indeed implies $h \to 0$.

Therefore, there are two families of singularities, each labeled by l, with $k = 2l$ ($l \geq 1$) and $k = 2l - 1$ ($l \geq 2$) respectively.[25] Choosing $N_c = l$, the singularity appears at a single point in the moduli space, which is $(l - 1)$-dimensional, so, as in the SYM case, one expects $l - 1$ relevant deformations of the IR SCFT, and this is what is found. For even k, these are the vevs of the $\frac{1}{2}(k - 2) = l - 1$ vector superfields in (4.12), while for odd k, these are the $[\frac{1}{2}(k - 2)] = l - 2$ vevs and one relevant coupling (that with $D(\mu) = 1/2$). The distinction between couplings and vevs indicates that, for even k, all the $l - 1$ massless vector superfields are interacting, while for odd k, one becomes free in the IR.

In addition, there are k mass parameters for the quarks in the gauge theory. As explained in subsection 4.1, these can be identified with vevs of the vector superfields in the Cartan subalgebra of the 6D $\mathrm{U}(k)$ gauge theory on the coinciding KK monopoles and these vector superfields are frozen in the 4D theory because of the infinite extent of the KK monopoles in two additional (internal) directions. These vector superfields couple to conserved (flavor) currents, so they should correspond to deformations with $D(\mu) = 1$. Strictly speaking, only the $k - 1$ non-diagonal flavor currents (those in $\mathrm{SU}(k)$) have the above interpretation. The deformations corresponding to these currents are those found above in the twisted sectors.

The diagonal flavor current has a somewhat more complicated nature. If the gauge group was $\mathrm{U}(N_c)$ instead of $\mathrm{SU}(N_c)$, the diagonal flavor current would couple to the $\mathrm{U}(1)$ factor. This is reflected also in the stringy embedding. In fact, the

[24] To be able to apply semi-classical consideration, as is done above, both scales should be sufficiently large, compared to the scale representing the $\mathrm{SU}(N_c)$ gauge coupling.

[25] The singularity with $k = 2l = 2$ is the same as the $l = 2$ singularity in SYM and, as there, it corresponds to SQED (see the end of the previous subsection).

original brane configuration, that with D4 branes, realizes classically a $\mathrm{U}(N_c)$ gauge theory, but it was argued in [45] that in this stringy realization, the vector superfield corresponding to the diagonal U(1) factor is frozen by quantum effects. In the configuration with NS5 branes and KK monopoles, the modulus of the U(1) factor in 6D $\mathrm{U}(k)$ gauge group corresponds to a collective motion of all the KK monopoles in the x direction, while the corresponding modulus in the 4D $\mathrm{U}(N_c)$ gauge group corresponds to such a motion of the NS5 brane. Since the quark masses depend on the relative displacement between the KK monopoles and the NS5 brane, the diagonal flavor current couples to a diagonal subgroup of the above two U(1) factors. For odd k, we did not find a deformation corresponding to this current. This suggests that it decouples from the IR SCFT. For even k, the corresponding deformation is the $D = 1$ deformation from the untwisted sector. Indeed, the deformation is $\delta W \sim x^{l-1} \sim \partial_x W$, which is an x-translation, corresponding to a relative motion between the NS5 brane and the KK monopoles.

Additional deformations in the field theory are moduli of the Higgs branch, corresponding to vevs of quark scalar fields. We do not find these in the worldsheet description. Their absence can be understood as follows:[26] at the singular point, to which the Higgs branch is connected, the quarks are massless. This corresponds, in the geometric formulation, to the singular hypersurface X^6, and in the worldsheet formulation, to a superpotential independent of the Liouville superfield $\hat{\phi}$. As explained in subsection 3.1, in this situation the worldsheet formulation suffers from a strong coupling singularity ($g_s \to \infty$ for $\phi \to -\infty$) and to avoid this singularity, one should consider the double scaling limit. In this limit, the mass m of the quarks is kept non-zero, which means that one actually considers a point in the Coulomb branch *near* the singular one. By taking the limit $m \to 0$, one can obtain information about the theory at the singularity, as was done above, but clearly this limit will miss aspects that appear discontinuously only at the singularity. This is why one does not see the Higgs moduli in this approach.

$\mathrm{SU}(l)$ **with $k = 2l$ massless quarks.** The $\mathrm{SU}(l)$ gauge theory with $k = 2l$ massless quarks is a conformally-invariant theory and the (complexified) gauge coupling τ is a modulus of the theory, which can be changed continuously. In particular, for $\tau \to i\infty$ one obtains a free field theory (at all scales). Therefore, unlike the other non trivial SCFT's, the one with $k = 2l$ is continuously related to a free theory, and the information implied by this relation can be compared to that obtained from the stringy embedding. This theory has an $\mathrm{SL}(2,\mathbb{Z})$ duality, relating different values of τ. The parameter g in (4.2) is an $\mathrm{SL}(2,\mathbb{Z})$-invariant function of τ (see [49, 50] for more details) and is, therefore, the quantity characterizing the strength of the interaction. Vanishing interaction corresponds to $g \to 0$, while for non vanishing g, the theory is an interacting SCFT.

[26] This explanation was suggested to the author by D. Kutasov.

The origin of the moduli space corresponds to $P(x) = x^l$ in (4.2), so $h = 1$ in (4.10). For $g = 1$, W takes the form

$$W = (z'^N - w'^N)^2 + uv\,,$$

which has a (non-isolated) $\mathbb{Z}_2$ singularity at $z'^N = w'^N$. The corresponding NS5 brane configuration includes two coinciding NS5 branes, which are the T-duals of the above $\mathbb{Z}_2$ singularity. Thus, in this case, the decoupled dynamics includes a six-dimensional sector.

To obtain an isolated singularity we, therefore, assume $g \neq 1$. In fact, we would like to consider $g \ll 1$, where one can apply semi-classical considerations. Then, the vector superfield $\mathcal{A}$ in the adjoint representation defines the following gauge-invariant vector superfields

$$\mathcal{U}_j = \mathrm{tr}\,\mathcal{A}^j\,, \qquad j = 2, \ldots, l$$

and their dimension is, classically (i.e., for $g \to 0$), $D(\mathcal{U}_j) = j$. The gauge coupling τ is identified as the coefficient of a term $\mathrm{tr}\mathcal{A}^2$ in the prepotential.

This information from the gauge theory agrees with the results obtained above from the stringy realization. The conformal dimensions found there were independent of the coupling, so should be the same as for $g \to 0$ and, indeed, the correct spectrum is obtained (see eq. (4.12)). The deformations $\delta H = \mu x^j$ correspond, for $j \leq l - 1$, to a vev $\mu \sim \mathrm{tr}\left\langle \mathcal{A}^{l-j} \right\rangle$ and for $j \geq l$, to a term $\mu\,\mathrm{tr}\,\mathcal{A}^{2+j-l}$ in the prepotential. In particular, for $j = l$ (corresponding to a term $\mu\,\mathrm{tr}\,\mathcal{A}^2$), μ is a change in the gauge coupling.[27] This can also be seen directly in the polynomial H (eq. (4.10) with $h = 1$): the corresponding deformation is $\delta H = \mu x^l$ and by rescaling w', this can be transformed to a change in g, which indeed represents the gauge coupling.

5. Summary

In this work, type-II string theory in a background of the form

$$\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times \mathrm{U}(1)_Y \times \mathcal{C} \tag{5.1}$$

was interpreted as a d-dimensional theory. For $\mathcal{C} = LG_W$ (a 2D $N = 2$ Landau-Ginzburg SCFT), this d-dimensional theory was identified in [3] as the description of the decoupled dynamics near an isolated singularity in type-II string theory on

$$\mathbb{R}^{d-1,1} \times X^{2n}\,, \tag{5.2}$$

where X^{2n} is a hypersurface $W = 0$ in a flat space $\mathbb{C}^{n+1}$ (with $d+2n = 10$). Here this identification was extended to $\mathcal{C} = LG_W/\Gamma$ (a Landau-Ginzburg orbifold), in which

[27] Note that such a marginal coupling (with $D(\mu) = 0$) does not appear in the other cases considered above, as expected from field theory.

case, X^{2n} is a hypersurface in an orbifold $\mathbb{C}^{n+1}/\Gamma$. Furthermore, for $d=4$, $\Gamma=\mathbb{Z}_k$ and special choices of the polynomial W, this four-dimensional theory was related to an interacting SCFT appearing in the moduli space of 4D $N=2$ SQCD: SU(N_c) gauge theory with fundamental quarks. Properties of the d dimensional theory were identified in the worldsheet formulation (of the string theory on (5.1)) and contrasted, when it was possible, with information from the geometric formulation (of string theory on (5.2); in section 3) and from the gauge theory (in section 4). Here we summarize this analysis.

We start with properties that are independent of the details of $\mathcal{C}$ and depend only on $\mathcal{C}$ being a 2D CFT with $(2,2)$ supersymmetry and a spectral flow operator relating the (c,c) and (a,a) rings. For any such $\mathcal{C}$:

- The theory has 2^{d72+1} supercharges; this corresponding to $N=2$ supersymmetry in 4 dimensions.

- It has a $\mathrm{U}(1)_+\times\mathrm{U}(1)_-$ R-symmetry, under which all the supercharges have charges $|R_\pm|=1$; in $d=4$, where the R-symmetry group is $\mathrm{U}(1)\times\mathrm{SU}(2)$, R_+ was identified with the U(1) factor and R_-, with a U(1) subgroup of the SU(2) factor; furthermore, in the 4D IR SCFT, R_+ was identified as the U(1) R-charge appearing in the superconformal algebra (SCA).[28]

- Each operator in the (c,c) ring of $\mathcal{C}$ defines a deformation of the theory, parametrized by a continuous parameter μ (which is a coupling of a term in the worldsheet lagrangian) with $R_-(\mu)=0$, and this deformation preserves the supersymmetry of the theory; in $d=4$, these deformations are related to scalars in vector multiplets (which are, indeed, invariant under the SU(2) R-symmetry): μ is either a vev (modulus) of a bottom component $\mathcal{A}_b$ of a vector superfield $\mathcal{A}$; or a coefficient (coupling) of a term in the 4D lagrangian, which is a top component $\mathcal{A}_t$ of such a superfield (corresponding to a term $\mu\mathcal{A}$ in the prepotential); this implies (using the SCA), that the conformal dimension of the deformation parameter is[28] (in both cases)

$$D(\mu)=\frac{1}{2}R_+(\mu)\,. \tag{5.3}$$

- The value of $R_+(\mu)$ is related to the ϕ-dependence of the corresponding worldsheet vertex operator (see eqs. (3.13) and (3.20)) and, consequently, provides information about the nature of the deformation: for $R_+(\mu)>2$, this is a modulus and for $R_+(\mu)<2$, this is a coupling; the coupling is relevant for $R_+(\mu)>0$, marginal for $R_+(\mu)=0$ and irrelevant for $R_+(\mu)<0$; in the IR

[28] Analogous identifications can be made also for $d\neq 4$.

CFT, the same information is provided by the conformal dimension of μ, where couplings are distinguished from moduli by using the representation theory of the SCA (including the "unitarity bound" $D(\mathcal{A}_b) \geq 1$ for a bottom components of a vector superfields $\mathcal{A}$); using the identification (5.3), these two approaches can be compared and one finds identical distinctions.

- Deformations with $R_+(\mu) \neq 2$, were shown to appear in pairs; for $d = 4$ this is a coupling-modulus pairing and this is identified with the pair of deformations that are defined by a given vector superfield.

Some of the evidence for the above identifications was obtained by considering specific cases, as described below. However, it is natural to expect that they have a more general range of validity.

We now turn to properties that are different for different $\mathcal{C}$ factors, corresponding to differences between d-dimensional theories (with $2^{d/2+1}$ supercharges). First, considering $\mathcal{C} = LG_W/\Gamma$ in general (including the case $\Gamma = 1$ studied in [3]), the above identifications can be compared with the geometric formulation of the theory, with the following results:

- One is led to the same requirements on the elements of Γ.
- One finds the same amount of supersymmetry: $2^{d/2+1}$ supercharges.
- The $\mathrm{U}(1)_+$ R-symmetry is identified as a geometric isometry.
- The (c, c) deformations from the untwisted sector are naturally identified as complex structure deformations, induced by changes in the polynomial W; this identification was shown to be consistent with the multiplicity and the R-charges of the deformation parameters.
- For the above deformations, a distinction between parameters and moduli can be identified also in the geometric formulation [28] and it was shown to agree with the distinction derived from the worldsheet formulation.
- Relevance of the above deformations translates, in the geometric description, to dominance of the change in W at $z \to 0$.

Finally, we considered specific four-dimensional SCFT's, appearing as IR limits of SQCD, in singular points of its moduli space. They are labeled by (k, l), where k is the number of massless quarks at the singularity and l is the degree of the singularity. We considered three families of these singularities, corresponding to $k = 0, 2l-1, 2l$, and chose $l = N_c$, (corresponding to the most singular point in the moduli space). For

JHEP03(2000)012

these theories, there is independent information, from field-theoretic considerations, and it was compared to the results form the worldsheet analysis:

- The holographic duality was shown to be a strong-weak coupling duality, in the sense that there is no situation in which both the gauge theory and the worldsheet CFT are weakly coupled.

- The effect of all the relevant and marginal (c, c) deformations (in both the untwisted and twisted sectors) was identified in the underlying gauge theory.

- The deformations expected in the gauge theory are $N_c - 1$ Coulomb moduli, k mass parameters (couplings to the flavor currents) and, for $k = 2l$, a gauge coupling.[29] The only deformation that was not identified is the coupling to the U(1) flavor current for $k = 2l - 1$. Its absence is an indication that this current decouples from the IR SCFT.

- In pure SYM ($k = 0$), field-theoretic considerations predict the number of interacting superfields (with $D > 1$), which are those coupled to mutually-non-local charges, and the number superfields (with $D = 1$) coupled to (non-trivial) mutually local charges. This prediction agrees with the worldsheet results. As to the other families of SCFT's, the conformal dimensions (4.12) indicate that, for $k = 2l$, all the $N_c - 1$ massless vector fields in the Coulomb branch are interacting and, for $k = 2l - 1$, one of them is free.

- For $k = 2l$, the SCFT is continuously connected to a free theory by changing the gauge coupling and the conformal dimensions were found to be independent of such a change. They should, therefore, be the same as in the free theory and indeed they were found to be so.

The above detailed agreement between results obtained using different approaches is a strong evidence for the duality proposed in [3] and in the present work. Additional evidence is found in the study of the $d = 6$ case, in [4, 5]. This duality can now be used to study the decoupled theory on the singularity, using the worldsheet formulation. In particular, one can calculate correlation functions of observables in the theory, as was done in [4, 5]. This will provide information about 4D $N = 2$ SCFT. It is also of interest to extend this duality further, e.g., to other gauge groups and to theories with less supersymmetry. All this is left for future study.

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[29]The moduli of the Higgs branch are not expected, as explained in subsection 4.4.

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JHEP03(2000)012

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